# Sine of natural numbers

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So, I want to prove that

$$\sup_{n\in\mathbb{N}} \sin(n) = 1$$

I was thinking of proving that some set related to $\pi$ is dense in $\mathbb{R}$ that will then imply there is some $n \in \mathbb{N}$ s.t. $\sin(n)$ is as close to 1 as desired. ($(\frac{m}{n}) \times \pi \quad \forall m, n \in \mathbb{N}$?)

## 3 Answers

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The additive subgroups of $\mathbb{R}$ are either dense or lattice (that is, $x\mathbb{Z}$ for some $x$). Since $\pi$ is irrational, the subgroup $\mathbb{Z} + 2\pi\mathbb{Z}$ is not lattice hence it is dense, in particular, for every $\varepsilon > 0$ there exists some integers $n$ and $m$ such that $|n + 2m\pi - \frac{1}{2}\pi| \leqslant \varepsilon$.

If $n \geqslant 0$, this yields $\sin(n) \geqslant \cos(\varepsilon)$. If $n < 0$, note that $|-3n - 6m\pi + 2\pi - \frac{1}{2}\pi| \leqslant 3\varepsilon$ hence $\sin(-3n) \geqslant \cos(3\varepsilon)$ with $-3n \geqslant 0$.

Since $\varepsilon > 0$ is arbitrary and $\cos(\varepsilon) \to 1$ and $\cos(3\varepsilon) \to 1$ when $\varepsilon \to 0$, this proves the result.

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Assume that isn't the case, i.e. $1 > \sup_{n\in\mathbb{N}} \sin(n) = 1 - \epsilon$ for some $\epsilon > 0$.

Let $\delta = \pi/2 - \arcsin(1-\epsilon)$. Then $|1 - \sin(n)| < \epsilon$ if $|n - (2\pi k + \pi/2)| < \delta$ for some $k \in \mathbb{N}$, which is equivalent to

$$|2\pi k - n| < \delta + \pi/2$$

It now follows from Dirichlet's approximation theorem that you can always find $n, k$ such that this condition is fulfilled. Contradiction!

*Edit: Actually, it seems that there should be an elementary justification for the last step since we don't even need the magnitude estimate from Dirichlet's theorem, but I can't think of one right now.*

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Another option using the equidistribution theorem, though this is something of a sledgehammer given the other answers: use the fact that $1/(2\pi)$ is irrational, so the sequence $n/(2\pi)$ for $n \in \mathbb{N}$ is equidistributed modulo 1. Therefore $n/(2\pi) \pmod 1$ visits points arbitrarily close to $1/4$ as $n \to \infty$, so $n \pmod{2\pi}$ visits points arbitrarily close to $\pi/2$, so $\sin n$ takes values arbitrarily close to $1$ as $n \to \infty$ (by continuity and periodicity of sin).